Lecture 8

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1 Matrix equations and the inverse

1.1 Discussion of the algorithm - Part 2

Last time we proved the following result:

Lemma 1.1. *1. If the square matrix A is invertible, then its RREF is the identity matrix.*

2. If we can reduce the matrix A by elementary row operations to the identity matrix, i.e. if its RREF is identity matrix, then this algorithm gives us $A^{-1}B$ in the right half of the augmented matrix.

Actually, the opposite assertion of this lemma is also true.

Lemma 1.2. *Let RREF of a square matrix A be the identity matrix. Then A is invertible.*

Proof. Let's consider the process of reducing A to it's RREF. It can be done by elementary row operations with matrices E_1, E_2, ..., E_s, i.e. $(E_sE_{s-1}\cdots E_1)A = I$. From this equality we see that the product of matrices $E_sE_{s-1}\cdots E_1$ satisfy the definition of the inverse for A. □

So, from these 2 lemmas we get the interesting result, which is the main result about invertible matrices so far:

Theorem 1.3. *The matrix A is invertible if and only if its RREF is the identity matrix.*

2 Vector spaces

In this lecture we will introduce a new algebraic structure which is one of the most important structure in linear algebra. This would be a set with 2 operations — addition of its elements and multiplication of numbers by its elements.

Definition 2.1. *Let $\Bbbk$ be any field. We didn't study fields so far, so those who are not familiar with them can just treat the letter $\Bbbk$ as another notation for $\mathbb{R}$. A set V is called* ***vector space*** *if there defined an operation of addition of elements of V such that $\forall v, w \in V \; v + w \in V$, and an operation of multiplication of elements of $\Bbbk$ by elements of V (often called* ***scalar multiplication****) such that $\forall k \in \Bbbk \; \forall v \in V \; kv \in V$, and the following axioms are satisfied:*
Axioms of addition:

(A1) $\forall v, u \in V \; v + u = u + v$

(A2) $\forall v, u, w \in V \; v + (u + w) = (v + u) + w$

(A3) $\exists \mathbf{0} \in V$ *such that* $v + \mathbf{0} = v$

(A4) $\forall v \in V \; \exists(-v) \in V$ *such that* $v + (-v) = \mathbf{0}$

Axioms of multiplication:

(M1) $\forall a \in \Bbbk \; \forall u, v \in V \; a(u + v) = au + av$

(M2) $\forall a, b \in \Bbbk \; \forall v \in V \; (a + b)v = av + bv$

(M3) $\forall a, b \in \Bbbk \; \forall v \in V \; a(bv) = (ab)v$

(M4) $\forall u \in V \; 1u = u$

Elements of the vector space are called ***vectors****.*

Now we'll give a number of examples of a vector spaces.

Example 2.2 (Space $\mathbb{R}^n$). *Let V be a set of n-tuples of elements of $\mathbb{R}$. We can define operations as follows:*

Addition: $(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$

Scalar multiplication: $k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$.

The zero vector is $\mathbf{0} = (0, 0, \ldots, 0)$ and the negative vector is $-(a_1, a_2, \ldots, a_n) = (-a_1, -a_2, \ldots, -a_n)$.

Example 2.3 (Space $P(t)$). *Let V be a set of all polynomials of the form*

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_s t^s, \quad s \in \mathbb{N}.$$

We can define operations as follows:

Addition: *Usual addition of polynomials.*

Scalar multiplication: *Multiplication of a polynomial by a number.*

The zero vector is $\mathbf{0} = 0$.

Example 2.4 (Space $P_n(t)$). *Let V be a set of all polynomials with degree less or equal to n of the form*

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_s t^s, \quad s \in \mathbb{N}, \quad s \leq n,$$

We can define operations as follows:

Addition: *Usual addition of polynomials.*

Scalar multiplication: *Multiplication of a polynomial by a number.*

The zero vector is $\mathbf{0} = 0$.

Example 2.5 (Space $M_{m,n}$). *Let V be a set of all $m \times n$-matrices. We can define operations as follows:*

Addition: *Usual addition of matrices.*

Scalar multiplication: *Multiplication of a matrix by a number.*

The zero vector is a matrix with all entries equal to 0.

Example 2.6 (Space $F(X)$). *Let V be a set of all functions from X to $\mathbb{R}$. We can define operations as follows:*

Addition: *Usual addition of functions:* $(f+g)(x) = f(x) + g(x)\ \forall x \in X$.

Scalar multiplication: *Multiplication of a function by a number:* $(kf)(x) = kf(x)\ \forall x \in X$.

The zero vector is a function $f(x) = 0\ \forall x \in X$. *The negative function is a function* $(-f)(x) = -f(x)\ \forall x \in X$.

And now we'll give an example of a set which is not a vector space.

Example 2.7. *Let's consider the polynomials of degree 10, i.e. set of functions $f(t)$ such that*

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{10} t^{10}.$$

Which axioms of a vector space does not hold here? First of all, this set doesn't have a zero element, since zero polynomial's degree is 0 — not 10. Moreover, we're not always able even to add polynomials, i.e. let's consider $f(t) = t^{10}$, and $g(t) = t^9 - t^{10}$. Degree of $f(t)$ and $g(t)$ are 10, but if we add them we'll get: $f(t) + g(t) = t^{10} + t^9 - t^{10} = t^9$ — and degree of the result is 9, not 10. So, a set of polynomials of degree 10 is not a vector space.

We can give some properties of vector spaces:

- If $u + w = v + w$ then $u = v$.

- $\forall k \in \Bbbk \quad k\mathbf{0} = \mathbf{0}$.

  Proof. $k\mathbf{0} = k(\mathbf{0} + \mathbf{0}) = k\mathbf{0} + k\mathbf{0}$, and so by the first property $\mathbf{0} = l\mathbf{0}$. □

- $\forall u \in V \quad 0u = \mathbf{0}$.

  Proof. $0u = (0 + 0)u = 0u + 0u$, and so by the first property $\mathbf{0} = 0u$. □

- If $k \neq 0$ and $ku = \mathbf{0}$ then $u = \mathbf{0}$.

  Proof. $u = 1u = (k^{-1}k)u = k^{-1}(ku) = k^{-1}\mathbf{0} = \mathbf{0}$. □

- $\forall k \in \Bbbk$ and $u \in V \quad (-k)u = k(-u)$.

  Proof. $\mathbf{0} = k\mathbf{0} = k(u + (-u)) = ku + k(-u)$, and $\mathbf{0} = 0u = (k + (-k))u = ku + (-k)u$. So, $k(-u) = (-k)u$. □

3 Subspaces

Definition 3.1. *Let V be a vector space. The subset $W \subset V$ is called a **subspace** of V if W itself is a vector space.*

To check that W is a subspace we need to check the following properties:

1. $\mathbf{0} \in W$
2. $\forall v, w \in W \quad v + w \in W$
3. $\forall k \in \Bbbk \;\; \forall u \in W \;\; ku \in W$

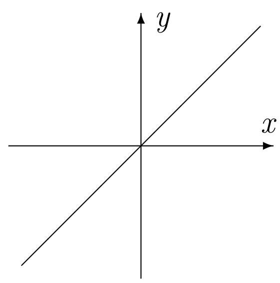
Example 3.2. *Consider a vector space $\mathbb{R}^2$. Then its subset $W = \{(0, y) | y \in \mathbb{R}\}$ — set of pairs for which the first element equals to 0, is a subspace.*

We can prove it. First of all, $(0, 0) \in W$, since first element of it is 0. Moreover, let $u = (0, a) \in W$, $v = (0, b) \in W$. Then their sum $u + v = (0, a + b) \in W$ since it has zero on the first place. Now let's multiply any vector $u = (0, a) \in W$ by any number k: $ku = (0, ka)$, and it belongs to W, since it has 0 on the first place. So, this is a subspace.

Example 3.3. *Consider a vector space* $\mathbb{R}^2$*. Then its subset* $W = \{(1, y) | y \in \mathbb{R}\}$ *— set of pairs for which the first element equals to 1, is NOT a subspace.*

Here the first property is not satisfied — $(0, 0)$ *doesn't belong to* W*. Other properties are not satisfied as well:* $(1, a) \in W$*,* $(1, b) \in W$*, but their sum* $(2, a + b) \notin W$*, since it has 2 on the first place.*

Example 3.4. *Consider a vector space* $\mathbb{R}^2$*. Then its subset* $W = \{(x, y) | x, y \in \mathbb{R},\ x = y\}$ *— set of pairs for which the first element is equal to the second element (geometrically, it is a line on the plane), is a subspace.*



Let's check it. First of all, if $\boldsymbol{a} = (a, a) \in W$*, and* $\boldsymbol{b} = (b, b) \in W$ *then* $\boldsymbol{a} + \boldsymbol{b} = (a + b, a + b) \in W$*. Than,* $(0, 0) \in W$*. Moreover, for each* $k \in \mathbb{R}$ *we have* $k(a, a) = (ka, ka) \in W$*. So, this is a subspace.*

One can prove that any line on the plane $\mathbb{R}^2$ which goes through the origin is a subspace. Moreover, any plane in the space $\mathbb{R}^3$ which contains the origin $(0, 0, 0)$ is a subspace.

Example 3.5. *Consider a vector space* $\mathbb{R}^2$*. Then its subset* $W = \{(x, x^2) | x \in \mathbb{R}\}$ *— set of pairs for which the second element is equal to the square of the first element is NOT a subspace.*

Let's prove it. First of all, if $(0, 0) = (0, 0^2) \in W$*. Now let's consider 2 elements of this set —* $(1, 1) \in W$ *and* $(2, 4) \in W$*. Their sum* $(3, 5)$ *doesn't belong to* W*, since* $5 \neq 3^2$*. So, we showed that there are two elements sum of which doesn't belong to the set. So, this is not a vector space.*